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ON THE PROBLEM OF THE STABILITY OF ONE-DIMENSIONAL UNBOUNDED ELASTIC SYSTEMS*

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An elastic, spring-supported beam along which a point mass is moving, is considered, and special features of the behaviour of such systems are pointed out. The stability of the system point mass-beam is studied. The velocity which, when exceeded, leads to instability of the beam is determined, and its dependence on the parameters of the system are studied.

The motion of an inhomogeneity through the system is accompanied by the generation of waves. The wave pattern can be separated into a stationary part containing "frozen" waves in a coordinate system attached to the body, and non-stationary waves, appearing in the course of transients or generated by the instability of the system in question.

Examples of the study of stationary waves are numerous, and appear in various branches of science. The study of non-stationary waves which can be used, in particular, in assessing the stability or instability of the body-medium systems, has received much less attention. Some fundamental problems related to the interaction of the body with a medium in relative motion, were solved in /1/.

The investigation of stability of the linear, homogeneous unbounded system usually begins with the dispersion equation, which gives the relation between the wave frequency and wave vector with real components /2/. The appearance of an inhomogeneity in the system makes it impossible to limit oneself to considering the dispersion equation only. One of the exceptional features of these problems is the lack of smoothness of the solution or of any of its derivatives at the point at which the inhomogeneity occurs, and the solution should be sought in the class of function vanishing at infinity (the components of the wave vector are complex). The necessity may also arise of making the model more complicated by e.g. introducing frictional forces.

Below, an example of an elastic, spring-supported beam is used to illustrate certain aspects of the behaviour of the unbounded medium interacting with a moving body. One-dimensional unbounded elastic systems were studied by a number of authors. The most interesting papers are /3, 4/ where, in particular, an instability was discovered caused by relative motion of the distributed mass of the pipe and a liquid flowing through it. In this connection the instability can be expected to appear also when a discrete mass moves along the elastic system (beam). It is natural to assume that the growing perturbation should concentrate near the position of the body whose oscillations impart energy to the beam, and should vanish (since the system is linear) at infinity.

We shall consider the following model. A point mass m moves along an infinite beam resting on an elastoviscous support. The motion of the mass consists of a motion with constant velocity v along the Ox axis, and of a motion along the Oy axis together with the beam (Fig.1) without separating from it. The behaviour of the beam is studied with reference to the $O\xi y$ system of coordinates moving along the Ox axis with velocity v, i.e. $\xi = x - vt$. The equation of beam flexure and the conditions for matching the solution at the position where the point mass appears, after changing to dimensionless coordinates, have the following form:

$$\frac{\partial^2 W}{\partial t^2} - 2v\frac{\partial^2 W}{\partial t\partial\xi} + v^2\frac{\partial^2 W}{\partial\xi^2} + \frac{\partial^4 W}{\partial\xi^4} - h\left(\frac{\partial W}{\partial t} - \frac{\partial W}{\partial\xi}v\right) + \frac{1}{4}W = 0 \qquad (1)$$

$$W_+(0, t) = W_-(0, t) \qquad (2)$$

$$\frac{\partial W_+(0, t)}{\partial\xi} = \frac{\partial W_-(0, t)}{\partial\xi}, \quad \frac{\partial^2 W_+(0, t)}{\partial\xi^2} = \frac{\partial^2 W_-(0, t)}{\partial\xi^2}$$

$$\frac{\partial^3 W_-(0, t)}{\partial\xi^3} - \frac{\partial^3 W_+(0, t)}{\partial\xi^3} = P + M\frac{\partial^2 W(0, t)}{\partial t^2}$$

$$\left(h = \eta\,(4d\rho)^{-1/2}, \quad P = mg\,(4dEI)^{-1/2}, \quad M = \frac{m}{\rho}\left(\frac{4d}{EI}\right)^{1/4}\right)$$

In addition, the solution of the problem must satisfy the condition of boundedness when $x \in (-\infty, \infty)$.

Here the time and length scales are given, respectively, by $(\rho\,(4d))^{1/2}$ and $(EI/(4d))^{1/4}$, EI is the flexural rigidity, and ρ is the mass per unit length of the beam. The parameter η characterizes the friction in the elastic support, and d describes its elastic properties, g

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is acceleration due to gravity, and $W_+(\xi, t)$ and $W_-(\xi, t)$ are the flexures of the beam to the left and right of the moving mass, respectively.

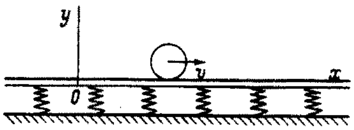

Fig.1

The solution of problem (1), (2) represents a superposition of the stationary $W_0(\xi)$ and non-stationary flexure, which we shall denote in what follows by $W(\xi, t)$. The stationary profile of the beam is described by the expression

$$W_{0\pm}(\xi) = e^{\mp a\xi}(A_{10}^{\pm}\cos b_{\pm}\xi + A_{20}^{\pm}\sin b_{\pm}\xi) \tag{3}$$
$$A_{10}^{+} = A_{10}^{-} = 4\frac{aP}{A_0}, \quad A_0 = 8a^2[2a^2 + (b_+ + b_-)^2] + (b_+^2 - b_-^2)^2$$
$$A_{20}^{+} = -P\frac{4a^2 - b_+^2 + b_-^2}{b_+A_0}, \quad A_{20}^{-} = P\frac{4a^2 + b_+^2 - b_-^2}{b_-A_0}$$

The quantities $a, b_{\pm}$ are given by the equation

$$\lambda^4 + v^2\lambda^2 - hv\lambda + 1/4 = 0 \tag{4}$$

whose real roots have the form $\lambda_{1,2} = -a \pm ib_+, \lambda_{3,4} = a \pm ib_-$.

When $h = 0$, we have $a = \left(\frac{1 - v^2}{4}\right)^{1/2}, \quad b_+ = b_- = \left(\frac{1 + v^2}{4}\right)^{1/2}$.

In the case when $v < 1$ the beam profile is determined by relations (3). As $v \to 1$ (the velocity approaches its critical value adopted here as the scale of velocity $v_k = (4dEI\rho^{-2})^{1/4}$ /5/, the beam flexure increases without limit. When $v > 1$, the problem has no solutions vanishing at infinity.

Both $W_{0+}(\xi)$ and $W_{0-}(\xi)$ represent the sum of four sinusoidal functions. To find all constants entering W_{0+} and W_{0-}, the condition that the solution merge at the point $\xi = 0$ is insufficient, i.e. the stationary flexure is not defined uniquely. To make the solution unique, we must change the formulation of the problem. We can, in particular, formulate certain supplementary conditions for $W_0(\xi)$. In the present case it is pertinent to use the principle of limit absorption /6/. To do this, we bring the friction into the discussion (damping at the support). The friction separates from the solution the terms decaying and growing at infinity, and we eliminate the latter. We can also obtain a unique solution for the conservative case by a passage to the limit $h \to 0$.

Fig.2 shows the stationary profile of the beam at various values of v. The sharp change (for small h) in the symmetric pattern of the stationary flexure at $v < 1$ to the asymmetric pattern at $v > 1$, merits attention. The calculations were carried out for $h = 0.05$. Note that similar graphs were obtained in /8/, though the beam stability was not discussed either in these, nor in any other papers familiar to the authors. Thus the question of realizing the stationary solutions obtained remains open.

Let us now consider the non-stationary solution of the problem (1), (2). The solution is sought in the form $W(\xi, t) = e^{pt - \lambda\xi}$. Substituting $W(\xi, t)$ into (1), we obtain

$$\lambda^4 + (p - v\lambda)^2 + h(p - v\lambda) + 1/4 = 0 \tag{5}$$

In general, the four complex roots λ_j ($j = 1, 2, 3, 4$) have at least one root with a positive, and one with negative real part, since the sum of the roots is zero by virtue of the fact that a term with λ^3 is missing from (5).

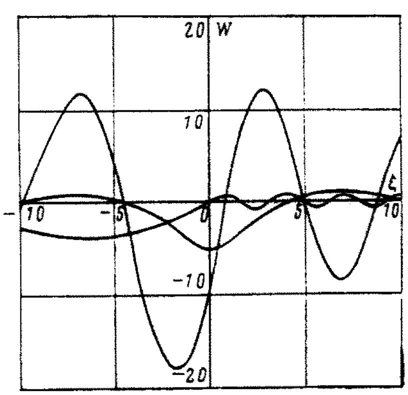

Fig.2

Three different versions of the distribution of the roots in the complex plane are possible, relative to the imaginary axis. In the first version the roots are on one side, in the second three roots are on the other side, and finally we have two roots in each half-plane.

Let us consider the last case, assuming, to be specific, that the real parts of the roots λ_1 and λ_2 are negative, and those of λ_3 and λ_4 are positive. The solution of (1) satisfying the condition for vanishing at infinity, takes the form

$$W_+(\xi, t) = e^{pt}(A_1e^{\lambda_1\xi} + A_2e^{\lambda_2\xi}); \quad W_-(\xi, t) = e^{pt}(B_1e^{\lambda_3\xi} + B_2e^{\lambda_4\xi})$$

By virtue of the linear character of the system, the load P does not appear, either in the equations of motion, or in the condition for matching the non-stationary solutions. The influence of the load on the non-stationary motions manifests itself in terms of the mass $M \neq 0$. When $M = 0$, the equations in deviations have no singularities when $\xi = 0$, therefore the elastic beam, regarded as a dissipative, load-free system, is stable.

The conditions for matching (2) lead to a homogeneous system of algebraic equations in A_1, A_2, B_1, B_2. Equating to zero the determinant of this system we obtain the "frequency" equation

$$\frac{2p^2(\lambda_1+\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)}\{M-\Lambda\}=0$$

$$\Lambda=\frac{(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}{2p^2(\lambda_1+\lambda_2)}$$

When $M \neq 0$, the spectrum of the eigenfrequencies of the system is mixed, and point frequencies are found from the condition

$$M=\Lambda \tag{6}$$

determining the stability of the system. Solving Eq.(5) under this condition, for specific values of the parameters appearing in it, we can determine the stability of the stationary profile of the beam under the perturbations caused by the moving mass, from the sign of the real part of the roots p. Such an approach, however, is cumbersome and not very effective.

In the present case the method of D-decomposition /9, 10/ is found to be suitable. Departing, for the time being, from the physical meaning of the problem, we shall regard M as a complex parameter and map the straight line $p=i\omega,\ \omega\in(-\infty,\infty)$ onto the complex plane M. We solve (5) for a certain value $\omega=\omega_0$, and having sorted out λ_j obtained in accordance with the signs of their real parts, we substitute them into condition (6) and then find $M(\omega_0)$. Here, as when determining the uniqueness of the stationary solution, we must take into account the friction $(h\neq 0)$, no matter how small, since it is only when it is present, that the roots λ_j at $p=i\omega$ acquire real parts and can be sorted out. By carrying out the calculations for different $\omega\in(-\infty,\infty)$, we obtain the curve $M(\omega)$ (the boundary of the D-decomposition) separating the complex plane M into regions with different numbers of characteristic indices p with positive real parts. When M have positive real values belonging to the region in which all $\operatorname{Re} p<0$, the system is stable. It becomes unstable, when the value of M is taken from the region in which at least one $\operatorname{Re} p>0$. From (5) and (6) it follows that the boundary of D-decomposition is symmetrical about the $\operatorname{Re} M$ axis.

The following cases are possible, depending on the value of the parameter v (Fig.3): $v<1$; the curve $M(\omega)$ does not intersect the $\operatorname{Re} M$ axis; $v>1$: the curve $M(\omega)$ intersects the $\operatorname{Re} M$ axis at one point $M_*(v)$. When v decreases, the point M_* moves to the right along the $\operatorname{Re} M$ axis and $M_*\to\infty$ as $v\to 1$. When the second parameter of the problem h increases, M_* also increases without affecting the qualitative behaviour of the curve $M(\omega)$.

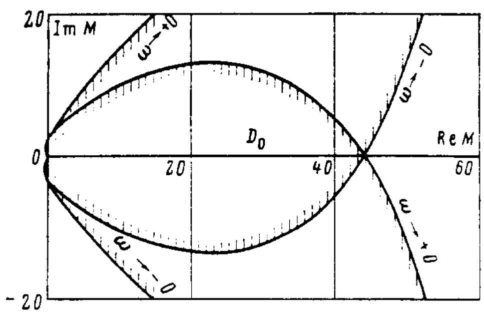

Fig.3

We have assumed in the above discussion a specific distribution of the roots of (5) in the complex plane, namely that of two roots on each side of the imaginary axis. In other cases, when a different possible distribution of the roots is obtained, Eq.(6) changes its form and becomes

$$M=-(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)p^{-2} \text{ for } \operatorname{Re}\lambda_1<0;\ \operatorname{Re}\lambda_2,\ \operatorname{Re}\lambda_3,\ \operatorname{Re}\lambda_4>0 \tag{7}$$

$$M=-(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)p^{-2} \text{ for } \operatorname{Re}\lambda_4>0;\ \operatorname{Re}\lambda_1,\ \operatorname{Re}\lambda_2,\ \operatorname{Re}\lambda_3<0$$

In the numerical program for computing the boundary of D-decomposition, we have taken into account all possible distributions of the roots λ_j, therefore the results discussed here are quite general.

Using the rule of hatching the boundary of D-decomposition (the hatching indicates the side of the boundary converted into a region with a larger number of roots with $\operatorname{Re} p<0$), we conclude that the number of roots with positive real parts is smallest in the region D_0 (Fig. 3), i.e. the range of values of $\operatorname{Re} M\ [0, M_*)$ belongs possibly to the domain of stability of the system.

We shall show that the stationary profile of the beam is stable for fairly low values of the mass M; therefore D_0 represents the domain of stability of the system.

With this purpose in mind, we shall use the equations of motion and the matching conditions in the coordinate system attached to the beam

$$\frac{\partial^2 W}{\partial t^2}-\frac{\partial^4 W}{\partial x^4}+h\frac{\partial W}{\partial t}-\frac{1}{4}W=0 \tag{8}$$

at the point $x=vt$

$$W_+(x=vt,t)=W_-(x=vt,t) \tag{9}$$

$$\frac{\partial W_+}{\partial x}=\frac{\partial W_-}{\partial x},\quad \frac{\partial^2 W_+}{\partial x^2}=\frac{\partial^2 W_-}{\partial x^2}$$

$$\frac{\partial^3 W_-}{\partial x^3}-\frac{\partial^3 W_+}{\partial x^3}=M\left(\frac{\partial^2 W}{\partial t^2}+2v\frac{\partial^2 W}{\partial x\,\partial t}+v^2\frac{\partial^2 W}{\partial x^2}\right)$$

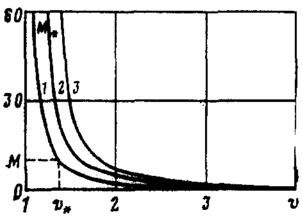

Fig.4

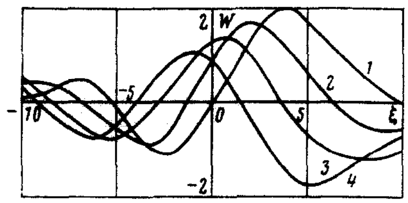

Fig.5

We assume that the function $W(x, t)$ is continuous on the set x, t together with its first and second derivative. The solution of the problem must decay at infinity, therefore, in what follows, we shall assume that $W(x, t)$ decreases as $x \to \pm\infty$ at least as rapidly as $\exp(-\alpha|x|)$ (α is a positive number.

In studying the stability we take the following functional as a measure of the perturbation (a prime and a dot denote differentiation with respect to x and t respectively, and a bar denotes a complex conjugate):

$$s(W, W^\cdot) = \int_{-\infty}^{\infty} [W''\bar{W}'' + W\bar{W} + W^\cdot\bar{W}^\cdot]\,dx + \sup_x W^\cdot\bar{W}^\cdot$$

and consider the positive definite functional

$$H(W, W^\cdot) = \int_{-\infty}^{\infty} \left[W''\bar{W}'' + \frac{1}{4}W\bar{W} + W^\cdot\bar{W}^\cdot\right]dx + MW^\cdot\bar{W}^\cdot|_{x=vt},$$

admitting an infinitesimal upper bound. The latter follows from the relation (M is a positive number)

$$H(W, W^\cdot) < \max\{M, 1\}\, s(W, W^\cdot)$$

At some fixed value of the parameter h there exists a finite neighbourhood of the point $M = 0$ at which the functional $H(W, W^\cdot)$ decreases with time by virtue of Eq.(8) and the conditions at the point $x = vt$ (9). Indeed, the condition

$$H^\cdot = -2h\int_{-\infty}^{\infty} W^\cdot\bar{W}^\cdot dx - vM\{W^\cdot(2\bar{W}^{\cdot\prime} + v\bar{W}'') + \bar{W}^\cdot(2W^{\cdot\prime} + vW'')\}|_{x=vt} < 0$$

clearly holds when the inequality

$$2h\int_{-\infty}^{\infty} W^\cdot\bar{W}^\cdot dx > vM\max\{|W^\cdot W^{\cdot\prime}|, v|W^\cdot W''|\}|_{x=vt} \qquad (10)$$

holds. The integral on the left-hand side of the inequality is positive, and bounded by virtue of the exponential decrease in the function $W(x, t)$ as $x \to \pm\infty$. The expressions within the braces are finite, since the functions in them are bounded for $x \in (-\infty, \infty)$.

Since the conditions of the theorem of the straight Lyapunov method on stability /11/ hold, we conclude that the system in question is stable for fairly small values of the parameter M. This, together with the D-decomposition of the complex M plane, leads to the final result: D_0 is the domain of stability, therefore the "stationary" profile of the beam is stable at values of M (regarded as a physical parameter) from the interval $[0, M_*)$. The conditions of stability (10) can easily be explained in physical terms. On the left we have a term reducing the energy of the system by virtue of the dissipation in the viscoelastic support, and on the right we have the term increasing the energy due to the oscillation of the mass M. When h increases while M and v decrease, the stability of the system increases.

We note that the inhomogeneity in the unbounded system can be caused not only by the point mass, but also by other factors (e.g. by an elastic point force). However, not every inhomogeneity arising during the motion along the beam can excite waves whose amplitudes increase with time. In particular, in the case of an elastic point force such wave excitation is not observed at any value of v. This can be shown using the direct Lyapunov method by just changing the form of the functional $H(W, W^\cdot)$.

Fig.4 shows the boundary 1, 2, 3 of the domain of stability in the mass-velocity plane for the values of the parameter $h = 0.1; 0.5; 0.9$. The domain of stability lies to the left of the curve $M(v)$ whose asymptotes are the straight line $v = 1$, and the Ov axis. At small values of the mass the system is stable over a wide range of velocities $[0, v_*)$. An increase in M is accompanied by a reduction in the domain of stability, and $v_* \to 1$.

Fig.5 shows the form of the oscillations of the beam at $M = M_*$, i.e. at the boundary of the domain of stability. The values $\omega t = 0, \pi/4, \pi/2, 3\pi/4$ correspond to the curves *1—4*. A wave appears near the mass *M*, moving in a direction opposite to thatof the mass. However, the directions of the motions of the mass nad the wave with respect to the beam are the same.

We note that the system in question can be used as a model of a pipe with a flow of fluid, made thicker at some place (increased mass), in the case when the ratio of the running mass of the pipe and the fluid is small. If the ratio is not small, then additional terms must be introduced in (1) /3/.

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AVERAGED DESCRIPTION OF THE OSCILLATIONS IN A ONE-DIMENSIONAL, RANDOMLY INHOMOGENEOUS MEDIUM*

A.YU. BELYAEV

The Cauchy problem for a wave equation with coefficients depending randomly on the spatial coordinate is considered. An equation describing the evolution of the expectation of the solution is derived assuming that the fluctuations of the coefficients and the correlation radius are small. The averaged equation, unlike the initial equation, is irreversible with respect to time, and has the form of a one-dimensional equation of motion of a viscoelastic material. The coefficient of effective viscosity obtained is found to be proportional to the intensity of fluctuations of the random characteristics of the inhomogeneous medium.

Numerous problems of the propagation of elastic, electromagentic and other waves in an inhomogeneous medium, reduce to solving the equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[a(x)\frac{\partial u}{\partial x}\right] \tag{1}$$

with initial data for $t = 0$. If the functions $\rho(x)$ and $a(x)$ characterizing the properties of the medium oscillate rapidly, then the problem arises of producing an averaged description of the wave propagation process. In randomly inhomogeneous continua the non-coherent character of wave dispersion by inhomogeneities of the medium produces a decay of solutions, which leads to the irreversibility of the averaged equations.

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